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External Noise and Its Interaction with Spatial Degrees of Freedom in Nonlinear Dissipative Systems

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A brief introduction to the field is given together with an overview of the lectures given at the workshop on External Noise and its Interaction with Spatial Degrees of Freedom in Nonlinear Dissipative Systems organized by the Center for Nonlinear Studies at Los Alamos, March 28–31, 1988. It is hoped that the publication of papers presented at the workshop in a single issue of the *Journal of Statistical Physics* will help draw attention to the recent developments in this rapidly area of nonequilibrium phenomena.

KEY WORDS: Stochastic processes: nonequilibrium phenomena; colored noise; multiplicative noise; nonlinear dynamics; transition to turbulence; spatial degrees of freedom; external noise; hydrodynamic instabilities; pattern formation; nonlinear optics; limits of computation.

1. MOTIVATION

In the last few years it has become increasingly clear from the study of non-equilibrium systems—i.e., of systems driven far from equilibrium by an external force such as a temperature gradient or a torque—that, in many cases, there is a strong interaction between spatial degrees of freedom and noise in spatially extended, large-aspect-ratio systems. Prominent examples of this interaction are open flow systems such as channel flow, shear layers, etc., for which the patterns (regular or turbulent) observed downstream

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depend crucially on the perturbations and on the noise acting on the system further upstream. Similar phenomena are also observed in confined flows which exhibit traveling wave instabilities. Other important examples of the interaction between noise and spatial degrees of freedom include pattern formation at the onset of thermal convection (a thin layer of fluid heated from below) and the pattern changes and transitions observed in the electrohydrodynamic instability of nematic liquid crystals under the influence of externally applied noise. In the latter case a sample of a nematic liquid crystal is sandwiched between conducting electrodes and an ac voltage is applied across the sample. Motivated by these experimental observations, it seems natural to consider the analytic and numerical tools which are available to describe, classify, and quantify the phenomena observed experimentally in noisy, spatially-extended systems. It was one of the major objectives of this workshop to bring together experimentalists and theorists working in these areas for the purpose of evaluating the current state of the art and to suggest future directions for research.

Another topic, which complements the one outlined above, is how the stochastic properties of a system change if the noise is no longer Gaussian and white. The limit of short correlation times is of particular interest. In an experimental system the noise is never perfectly Gaussian and white, but nevertheless frequently has a correlation time which is short compared to all characteristic macroscopic time scales of the system. In other words, the noise can be considered Gaussian and white to a very good degree of approximation. Of particular importance is whether the limit of Gaussian white noise is obtained smoothly as the correlation time goes to zero or when coming from a finite correlation time, or whether the limit is singular. It is now customary to call noise which is nonwhite, whether Gaussian or not, "colored noise" and it was a goal of this workshop to evaluate the state of the art in this field. This seemed all the more important since various groups had suggested a number of different approaches leading to results which seemed incompatible and this in turn sparked a number of controversies in the literature. Having all the major opponents at a single meeting helped to clarify things, at least in the most important practical case of Gaussian noise with a short correlation time.

Finally, it seemed important to clarify the situation in the field of stochastic processes with respect to other recently developing areas, such as the study of chaos, a term which characterizes irregular deterministic behavior. It is natural to ask how a truly random phenomenon can be distinguished from chaotic behavior. Other questions in this general area include the possible relationship with $1/f$ noise, the limits of computation and its sensitivity to noise, the difference in the stochastic properties of classical versus quantum noise as they occur in nonlinear optics, and also whether there is any relationship between stochastic processes and phenomena such as self-organized critically, which have been observed in cellular automata and have possible applications to physical systems.

The collection of papers at the workshop therefore presents itself naturally in three groups.

1. Experiments and theories describing the interaction between spatial degrees of freedom and noise.
2. Colored noise: theory and simulations.
3. General theoretical aspects and connections with other fields.

The introduction is concluded with a brief perspective.

2. EXPERIMENTS AND THEORIES DESCRIBING THE INTERACTION SPATIAL DEGREES OF FREEDOM AND NOISE

In the paper presented by G. Ahlers, deterministic and stochastic effects near the onset of thermal convection were studied using a heat current which is ramped linearly in time, or alternatively a heat current which has a constant piece and a temporally-modulated piece. The patterns arising at onset of convection for the linear ramp are irreproducible and irregular, which suggests that stochastic effects are dominant. The noise strength necessary to fit the experimental data is, however, four orders of magnitiude larger than that expected for thermal noise. In the case of temporal modulation one obtains regular and reproducible patterns near onset for large constant driving, whereas for smaller driving it is irregular and irreproducible, just as for the linear ramp.

Kai *et al.* investigated the influence on pattern selection of external noise applied to the driving voltage in the electrohydrodynamic instability of nematic liquid crystals. This a particularly convenient system for experimental investigations involving noise, since it can be easily added to the driving voltage in a well-controlled way. In addition, it is easy to realize large aspect ratios for EHD in nematics. The crucial parameters are the noise strength and the correlation time of the noise, whereas there is only a weak dependence on the type of noise, as long as the correlation time is sufficiently short. In this case one observes an upward curvature, with increasing noise intensity, of the threshold voltage plotted as a function of noise intensity, and jumps in the curvature when the first instability occurring at onset changes in nature—e.g., from fluctuating Williams domains to the grid pattern. It is also worth noting that the nonlinear onset time for the patterns depends strongly on the noise intensity, and that the corresponding relaxation rate shows a linear decrease for the spatially regular patterns as predicted for simple nonlinear models which have been solved exactly. Finally, it should be stressed that for large noise correlation time a destablization by noise is obtained, a feature not predicted by any of the currently available theoretical models.

J. Tough gave an overview of his experiments on the influence of noise on the transition to superfluid turbulence under an external heat flow in a thin capillary. The interpretation of these experiments is complicated because there is no information available about the actual spatial pattern—i.e., the spatial distribution of vortices is unknown. This situation is somewhat reminiscent of older studies of thermal convection in normal fluid ^{4}He, which also suffered a lack of flow visualization. It is found that the turbulent regime undergoes a continuous transition (TI–TII transition) with increasing heat current; at the transition intrinsic fluctuations in the dissipation and the relaxation time both become large. Many features experimentally observed are compatible with a model describing the transition as an imperfect pitchfork bifurcation without spatial degrees of freedom (due to Horsthemke and Schumaker). Weak external noise added to the driving heat current only results in fluctuations about the steady state. For strong external noise, however, a large change arises and the coexistence of two different turbulent states is observed, i.e., a noise-induced transition to bistability is observed. W. Horsthemke gave a general overview of the modeling of the TI–TII superfluid turbulence transition studied experimentally by Tough. For intrinsic noise the model using an imperfect pitchfork bifurcation gives a satisfactory description of the experimental data. For the case of strong external noise a first phenomenological model was suggested, which contains as a special case the model for intrinsic and weak external noise. Global properties are important for strong external noise, responsible for the coexistence of different states. This model is explained in detail in the paper of Schumaker and Horsthemke.

In their experiments on a small-aspect-ratio Bénard convection cell containing a dilute superfluid ^{3}He/^{4}He superfluid mixture, R. Ecke and H. Haucke investigated the influence of noise on the quasiperiodic regime. They found that sufficiently close to but below the chaotic regime, a small amount of noise superposed on the cooling current can induce the transition from quasiperiodicity to intermittency. This transition induced by external multiplicative noise is understood by making the observation that noise can cause the system to escape the weakly attracting periodic points.

R. J. Deissler presented an overview of his work on the complex Ginzburg–Landau equation with complex coefficients, a convective term, and additive fluctuations. He stressed the importance of the concept of convective instability (a perturbation decays when observed at the same spot, but grows when following it downstream) versus absolute instability (a perturbation grows when observed at the same spot) for open flow systems with a through flux and for confined systems with traveling waves. He showed several computer movies of the results of his simulations and suggested the term "noise-sustained structure" for the downstream structure that moves out if the upstream perturbation is removed. This picture holds for forward bifurcations. If the bifurcation is weakly inverted as is the case for several experimentally accessible systems, localized objects (slugs) form which are followed by laminar regions, i.e., one has the "coexistence of two different attractors."

G. Mayer-Kress discussed how one can characterize the dynamic behavior of spatially extended systems in their turbulent and regular regions. The model used consisted of coupled map lattices (as an example the logistic map was taken) which show spatiotemporal chaos. Correlation and Lyapunov dimension densities were used to analyze the data and the results were compared with the case of purely stochastic behavior. It also emerged from these studies that external noise can induce transitions between different patterns.

Taking a more analytic approach, B. West reported a study of the conditions for macroscopic segregation of two species in a simple chemical reaction subject to random particle inputs. The major result is that, for equal *global* concentrations of both species, there is segregation in one and two dimensions. In three dimensions, the marginal one, segregation depends on the parameter values of the model.

3. COLORED NOISE: THEORY AND SIMULATIONS

Starting with a Langevin-like equation, Van Kampen considered a stochastic force which is a stationary Gaussian stochastic process with zero mean. He then derived a Fokker–Planck-like equation which contained as a special case the limit of Gaussian white noise. He emphasized that this is a double expansion; first, a cumulant expansion of the stochastic differential equation (an expansion in the product of the noise intensity and the correlation time of the noise) and, second, an expansion of the coefficient of the second derivative in the Fokker–Planck-like equation in powers of the correlation time. Taking account of higher derivatives, one also has an expansion in the gradients. For non-Markovian time-dependent behavior, van Kampen stressed the delicacy of first-passage time problems.

The question of the time-dependent behavior was further elucidated by Hagan *et al.*, who discussed the exit time (first passage time) problem for the Ornstein–Uhlenbeck model and determined asymptotically the exit time probability distribution. To achieve this, a Laplace transform is taken and then singular perturbation methods are used to reduce the Fokker–Planck equation in two variables to a boundary layer problem. Hänggi *et al.* attacked the time-dependent problem from a different perspective. They discussed the bistable dynamics for a system with colored noise. Both the stationary behavior and the relaxation from a metastable state were considered, and numerical results were given for all values of the correlation time. The stationary probability density and the escape rate at large correlation time are compared with those of an approximate theory. Kłosek-Dygas *et al.* investigated the time-dependent problem from another analytical point of view. They considered the influence of colored noise on activation rates and emphasized that one is expanding in two small parameters, the correlation time of the noise and the noise intensity. Depending on the relative magnitude of these two small quantities—noise intensity and correlation time—three different cases arise and were discussed in detail. The accuracy of numerical mean-first-passage-time calculations was a central issue in the presentation by Fox. He described the numerical simulation of stochastic differential equations and stressed the importance of using simple but highly accurate algorithms. He reported excellent agreement between his own numerical results and mean-first-passage-time calculations carried out analytically by Hagan, Doering, and Levermore.

The section on numerical schemes was continued by Moss, who described analogue simulations of a bistable system driven by white or colored noise, with a spatially varying noise intensity. He showed that the most probable state of the system can be reversed by altering the noise intensity in the neighborhood of the barrier, thus supporting previous analytic work by Landauer and van Kampen. In the presentation by Debnath *et al.* it was demonstrated that the probability density appearing for systems driven by strongly colored noise can acquire holes. This was shown explicitly to occur for bistable systems using analogue simulations, analytic theory, and matrix continued fraction methods. For small noise correlation time one has a pair of local maxima, whereas for sufficiently large correlation time the single saddle disappears and is replaced by a pair of off-axis saddles. As a consequence, a hole bounded by the saddles and the local maxima arises. Analogue simulations were also reported for the influence of multiplicative colored noise on the equation for the magnetic Fredericksz transition in nematic liquid crystals. Dichotomous noise and the physically more realistic Gaussian distributed noise were investigated. The results obtained for the two different types of noise were quite different. Unfortunately no experimental results are available to verify these predictions in a liquid-crystal system.

4. GENERAL THEORETICAL ASPECTS AND CONNECTIONS WITH OTHER FIELDS

The transition from the field of colored noise to the more general topics covered in the workshop was naturally provided by the talk of R. Roy, who presented measurements of power spectra of a single mode in a ring dye laser demonstrating a drastic increase in the relaxation time close the threshold. The experimental results obtained were shown to be in qualitative agreement with numerical simulations and analytic calculations based on a simple models including multiplicative noise. In addition, the dynamics of the transition between various transverse mode patterns was investigated using first-passage-time distributions. In the second paper dealing with optics, A. Schenzle gave a review of the importance of classical and quantum noise in a variety of nonlinear optical systems. Analytically soluble models were presented to describe the stationary and time-dependent behavior of the dye laser, one with loss and one with gain noise, both of which give qualitative agreement with experimental results. The importance of quantum noise was demonstrated for the example of quantum jumps, which can now be observed experimentally, since it is possible to trap a single atom in a cavity.

The same approach, namely a Fokker–Planck equation for Gaussian white noise, was used by R. Graham as the starting point for the investigation of bifurcations under weak Gaussian white noise. In the weak noise limit the Fokker–Planck equation can be transformed into a Hamilton–Jacobi-type equation, and it was shown that the corresponding dynamical systems share some general asymptotic properties similar to those known from equilibrium thermodynamics. This was explicitly demonstrated for the examples of a codimension-two bifurcation with double eigenvalue zero and for the noise in a pendulum with an external torque. The influence of noise on bifurcations was further examined by Sigeti and Horsthemke, who showed that the external noise applied in the vicinity of a saddle-node bifurcation, which shows relaxation oscillations, leads to noise-induced oscillations with a period which has a well-defined most probable value. The corresponding noise-induced frequency—in a system with no intrinsic deterministic time scale—shows up as a peak in the power spectrum, whose location depends on the strength of the applied noise. This is a remarkable prediction which should be fairly straightforward to test experimentally.

A similar evaluation applies to an effect considered by M. Büttiker. He considered overdamped particles subject to drift in a force field with sinusoidal spatial dependence and similar dependence of the diffusion coefficient (with the same period). A noise-induced current results, depending on the amplitude of the diffusion modulation and on the phase difference between drift and diffusion. In the next paper by M. Büttiker and H. Thomas, the spatial dependence enters through derivative terms which arise when one examines the propagation of kinks in driven and damped nonlinear Klein–Gordon chains. One finds that the propagation velocity of the kinks as a function of the driving field and the kink width as a function of the propagation velocity depend only on the form of the potential. In addition to the anticipated discrete localized eigenmodes, a two-dimensional continuum of oscillatory modes with localized envelopes emerges.

The spatial extent of the system enters in a much more dominant way in the contribution presented by K. Wiesenfeld. He showed that there are spatially extended dynamical systems which evolve toward a state characterized by domains of all length scales. This phenomenon is called self-organized criticality, to emphasize similarities with equilibrium phase transitions. Explicit results were given for a cellular automaton modeling a sandpile undergoing avalanches.

The connection between stochastic processes and quantum mechanics was addressed by P. V. E. McClintock. He discussed the possibility of performing analogue simulations of quantum mechanical systems. Specifically, he chose as an example the quantum harmonic osciallator, whose properties are known exactly. This method exploits the relationship between the Fokker–Planck equation for Gaussian white noise and the Schrödinger equation; the respective potentials are related by a Ricatti-type equation which needs to be solved digitally. This is an interesting new way of looking at an old problem. Its future impact will depend on whether it is more efficient to do an analogue simulation with an interwoven digital step than to solve the underlying Schrödinger equation using only digital methods. This is especially the case for nonlinear potentials and for several variables, for which the resulting Ricatti-type equation can be difficult to solve.

This question of the efficiency of analogue versus digital computations brings us naturally to the presentation of R. Landauer, who argued that computation can be carried out with arbitrary little dissipation per step, provided it is done sufficiently slowly; only the information-discarding step sets a lower bound on the dissipation. This has to be contrasted with the situation in communication, where the latter type of step is not required.

5. CONCLUSIONS AND PERSPECTIVE

Several key features appeared as a result of the workshop. First, it became clear that for the description of the interaction between noise and spatial degrees of freedom, no efficient, general, analytic approaches are yet available for realistic systems. The concepts of Lyapunov and correlation dimension densities have been developed for coupled map lattices, but need to be implemented for partial differential equations as well. On the experimental side one would like to see more quantitative experiments on a variety of systems—for which the patterns must be visible—in order to have a large number of scenarios on which one can then hope to build up one's intuition for these complex systems. Quite remarkably, however, even very simple questions remain unanswered: the additive noise at the convective onset is four orders of magnitude larger than that expected on thermodynamic grounds.

For the problem of colored noise a substantial clarification has been achieved for the case of exponentially-correlated Gaussian noise with short correlation times. All theoretical studies agree in the white noise limit, but the approach to this limit—especially for first-passage times—is a delicate issue, depending on the order in which limits are taken, and in a non-analytic way on the correlation time. For larger correlation times and for the general time-dependent problem there is no generally accepted approach available, even without considering non-Gaussian noise. In this area the problem clearly boils down to how to proceed beyond an effective Fokker–Planck equation and to obtain, nevertheless, a tractable equation for the probability density—a challenge left to future research.

It also becomes clear that there are now a few tools available to distinguish chaotic and stochastic behavior, e.g., from time series. These include correlation and Lyapunov dimensions and specific features in the power spectra. It seems most important at this point in this area to find an experimentally acessible system for which both experimental and theoretical results can be compared quantitatively.

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